Permanent income hypothesis: a formal statement.

Time is discrete and horizon is infinite. Assume that there is an infinitely-lived individual who faces an exogenous deterministic stream of income $\{y_t\}_{t=0}^{\infty}$ and an exogenous deterministic sequence of (real) interest rates $\{r_t\}_{t=0}^{\infty}$. The agent seeks to maximize her discounted life-time utility, where one-period utility function is $u(c)$, and the discount factor is β, $0 < \beta < 1$. The sequence problem of the agent is, thus,

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{1}$$

subject to the sequence (one for each t) budget constraints,

$$\frac{A_{t+1}}{1 + r_{t+1}} = A_t + y_t - c_t, \tag{2}$$

where A_t is agent's wealth at time t and A_0 is given. The budget constraint states that the present value of next-period wealth, $\frac{A_{t+1}}{1+r_{t+1}}$, is the current resources, $A_t + y_t$, net of current consumption, c_t. Notice that we implicitly assume the existence of a perfect asset market which allows the agent to borrow/lend at next-period interest rate, r_{t+1}.

Here A_t is the state variable of the agent (the agents starts period t having wealth A_t) and A_{t+1} and c_t are control variables (in each period agent chooses how much to consume and how much wealth to set aside for the future).The Bellman equation associated with this problem is

$$V(A_t) = \max \{u(c_t) + \beta V(A_{t+1})\} \tag{3}$$

subject to the (time-t) budget constraint

$$A_{t+1} = (1 + r_{t+1}) (A_t + y_t - c_t). \tag{4}$$

One can use (4) to eliminate consumption in the Bellman equation (3), which then reduces to:

$$V(A_t) = \max \left\{ u \left(A_t + y_t - \frac{A_{t+1}}{1 + r_{t+1}} \right) + \beta V(A_{t+1}) \right\}. \tag{5}$$

The first order condition for (5) is:

$$-u'(c_t)\frac{1}{1+r_{t+1}} + \beta V'(A_{t+1}) = 0, \tag{6}$$

and the envelope condition is:

$$V'(A_t) = u'(c_t).$$

Updating the latter by one period and substituting the result in (6), one obtains the Euler equation:

$$u'(c_t) = \beta\,(1 + r_{t+1})\,u'(c_{t+1}), \tag{7}$$

which describes the evolution of consumption along the optimal path. As is known, the Euler equation is only a necessary condition for optimality; the necessary and sufficient conditions are the Euler equation and the transversality condition, which in this case writes as:

$$\lim \beta^t u'(c_t) A_t = 0. \tag{8}$$

Let us now start making additional assumptions. The first one is that I assume that the interest rate r is constant and is equal to the discount rate, $r_t \equiv r = \frac{1}{\beta} - 1$. This implies (by virtue of the Euler equation 7) that consumption is constant over time, $c_t = c$.

Given A_0, one can use the sequence of the budget constraints (4) to recover c. In period zero, the budget constraint is:

$$A_0 = \frac{A_1}{1+r} - (y_0 - c);$$

in period one, the constraint is:

$$A_1 = \frac{A_2}{1+r} - (y_1 - c),$$

which implies:

$$A_0 = \frac{1}{1+r}\left(\frac{A_2}{1+r} - (y_1 - c)\right) - (y_0 - c).$$

Continuing by induction, one obtains,

$$A_0 = \left(\frac{1}{1+r}\right)^t A_t - \sum_{k=0}^{t-1}\left(\frac{1}{1+r}\right)^k (y_k - c). \tag{9}$$

The transversality condition (8) implies that in the limit as $t \to \infty$, the first term in the right-hand side of (9) vanishes, so that:

$$A_0 = -\sum_{k=0}^{\infty} \left(\frac{1}{1+r}\right)^k (y_k - c). \tag{10}$$

The latter can be rearranged to yield,

$$c = \frac{r}{1+r} \left(A_0 + \sum_{k=0}^{\infty} \left(\frac{1}{1+r}\right)^k y_k \right), \tag{11}$$

so that consumption is the annuity (annual rent) value of future income. We shall term the latter a *permanent income*, thus making no difference between the permanent income theory of Friedman (1957) and the life-cycle theory of Modigliani and Brumberg (1954).

Now let us see what changes if a more realistic assumption of income uncertainty is brought in, i.e. from now on $\{y_t\}_{t=0}^{\infty}$ is a given sequence of random variables. (Recall that I still maintain the assumption about the interest rate being equal to the discount rate.) The problem of the agent is now to maximize *expected* discounted utility of future consumption, that is to maximize

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{12}$$

subject to the same sequence of budget constraints, where E_0 denotes expectation as of time zero. Furthermore, following Hall (1978), assume that agents form rational expectations, so that E_0 is the mathematical expectations operator. Notice that tautologically I can write the sequence problem beginning from any pre-specified time period t. Now the Bellman equation will have an expectation operator in it, so that

$$V(A_t) = \max \left\{ u\left(A_t + y_t - \frac{A_{t+1}}{1+r_{t+1}}\right) + \beta E_t V(A_{t+1}) \right\}, \tag{13}$$

and hence, the Euler equation is:

$$u'(c_t) = \beta E_t \left[(1 + r_{t+1})\, u'(c_{t+1}) \right], \tag{14}$$

or taking into account that $r_t = \frac{1}{\beta} - 1$ for all t,

$$u'(c_t) = E_t u'(c_{t+1}). \tag{15}$$

Next, assume that $u(c)$ is quadratic, so that marginal utility, $u'(c)$ is linear, $u'(c) = \gamma_0 - \gamma_1 c$. Then, (15) simplifies to:

$$c_t = E_t c_{t+1}, \tag{16}$$

so that consumption follows a martingale. The martingale property implies that consumption can be written as:

$$c_{t+1} = c_t + \epsilon_t,$$

where ϵ_t has zero mean for all t, but ϵ_t are not necessarily identically independently distributed. In other words, consumption can not be predicted from lagged consumption, contrary to the standard (at that time — mid-seventies) specification of consumption functions.

Now let us work a bit with the budget constraints. Take (10), which now must hold in expectation,

$$A_0 = -E_0 \left[\sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^k (y_k - c_k) \right], \tag{17}$$

and notice that a similar constraint must hold for an arbitrary t,

$$A_t = -E_t \left[\sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^k (y_{t+k} - c_{t+k}) \right]. \tag{18}$$

The martingale property of consumption implies that:

$$E_t c_{t+k} = E_t c_{t+1} = c_t$$

for all k, which allows to rewrite (18) as:

$$c_t = \frac{r}{1+r} \left(A_t + \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^k E_t y_{t+k} \right) \tag{19}$$

— a stochastic version of (11). To arrive at our final expression, lag (19) by one period

$$c_{t-1} = \frac{r}{1+r} \left(A_{t-1} + y_{t-1} + \frac{1}{1+r} \sum_{k=0}^{\infty} \left(\frac{1}{1+r} \right)^k E_{t-1} y_{t+k} \right) \tag{20}$$

and use the budget constraint,

$$A_t = (1+r)(A_{t-1} + y_{t-1} - c_{t-1}),$$

to get rid of A_t in (19). Then (19) and (20) can be written as (respectively):

$$c_t = r(A_{t-1} + y_{t-1} - c_{t-1}) + \frac{r}{1+r}\sum_{k=0}^{\infty}\left(\frac{1}{1+r}\right)^k E_t y_{t+k} \tag{21}$$

and

$$(1+r)c_{t-1} = r(A_{t-1} + y_{t-1}) + \frac{r}{1+r}\sum_{k=0}^{\infty}\left(\frac{1}{1+r}\right)^k E_{t-1} y_{t+k}$$

or

$$c_{t-1} = r(A_{t-1} + y_{t-1} - c_{t-1}) + \frac{r}{1+r}\sum_{k=0}^{\infty}\left(\frac{1}{1+r}\right)^k E_{t-1} y_{t+k}. \tag{22}$$

subtraction of (22) from (21) yields:

$$\Delta c_t = \frac{r}{1+r}\sum_{k=0}^{\infty}\left(\frac{1}{1+r}\right)^k \left(E_t - E_{t-1}\right) y_{t+k}, \tag{23}$$

which is our final statement of the permanent income hypothesis.

Now, what's the big deal of that? Its implications. Look, the change in consumption is proportional to the current innovation to income and cannot be predicted by regressing consumption on lagged consumption and/or income. This is in sharp contrast with a postulate of an existing stable relationship between consumption and lagged income — a routine assumption used in the seventies to estimate consumption functions. Indeed, assume as it was standard at that time, that

$$c_t = f(y_t, A_t, \Gamma(L)y_t),$$

where $\Gamma(L)$ is some lag polynomial. The function f can be taken linear without loss of generality because any non-linear function can be linearized in the neighborhood of the observed mean of its arguments,

$$c_t = c + \gamma A_t + \alpha_0 y_t + \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + ... + \varepsilon_t. \tag{24}$$

If that is a true stable relationship between consumption and income, then it survives being put to the first differences:

$$\Delta c_t = \gamma \Delta A_t + \alpha_0 \Delta y_t + \alpha_1 \Delta y_{t-1} + \alpha_2 \Delta y_{t-2} + ... + \varepsilon_t - \varepsilon_{t-1}.$$

The permanent income hypothesis (see (23) above) implies that

$$\gamma = 0 \text{ and } \alpha_i = 0 \text{ for } i = 0, 1, ...$$

or else (24) is inconsistent with (23). This prediction was a big shock to many of Hall's colleagues for it trashed a good deal their research on consumption.

The implication of the permanent income hypothesis that consumption is independent of past income is often called an *orthogonality proposition.* Hall (1978) brought this proposition to data and found that although consumption is orthogonal to past income, it is not orthogonal to lagged asset holdings, which gave him the grounds for rejection of the hypothesis. Nevertheless, the bomb has exploded, and his paper changed the research agenda on consumption forever.

The implication of (23) is not only orthogonality of consumption, but also a quantitative implication about how much volatile consumption should be. Had we known the process, which generates income, then we could test the validity of Hall's proposition based on the observed volatility of consumption. To illustrate the point, assume that income follows a simple MA(1) process:

$$y_t = \mu + \varepsilon_t + \beta\varepsilon_{t-1}.$$

Then,

$$(E_t - E_{t-1})\, y_t = \underbrace{\mu + \varepsilon_t + \beta\varepsilon_{t-1}}_{E_t y_t} - \underbrace{(\mu + \beta\varepsilon_{t-1})}_{E_{t-1} y_t} = \varepsilon_t,$$

$$(E_t - E_{t-1})\, y_{t+1} = \underbrace{\mu + \beta\varepsilon_t}_{E_t y_{t+1}} - \underbrace{\mu}_{E_{t-1} y_{t+1}} = \beta\varepsilon_t,$$

$$(E_t - E_{t-1})\, y_{t+k} = \underbrace{\mu}_{E_t y_{t+k}} - \underbrace{\mu}_{E_{t-1} y_{t+k}} = 0 \text{ for all } k \geq 2.$$

Thus,

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0} \left(1 + \frac{\beta}{1+r}\right) \varepsilon_t.$$

For a general MA(n) process for income,

$$y_t = \mu + \varepsilon_t + \beta_1\varepsilon_{t-1} + \beta_2\varepsilon_{t-2} + ... + \beta_n\varepsilon_{t-n},$$

$$\Delta c_t = \frac{r}{1+r} \sum_{k=0}^{\infty} \left(1 + \frac{\beta_1}{1+r} + \frac{\beta_2}{(1+r)^2} + ... + \frac{\beta_n}{(1+r)^n}\right) \varepsilon_t,$$

where n can be infinity. The case of an infinite MA process is particularly important because, according to Wold theorem, any stationary AR(n) process can be represented as an infinite MA process. Notice, that for a general ARMA process for income,

$$D(L)y_t = B(L)\varepsilon_t,$$

(23) can be written as

$$\Delta c_t = \frac{r}{1+r}\frac{B\left(\frac{1}{1+r}\right)}{D\left(\frac{1}{1+r}\right)}\varepsilon_t, \tag{25}$$

where $B\left(\frac{1}{1+r}\right)$ is the lag polynomial

$$B(L) = \mu + \beta_1 L + \beta_2 L^2 + ...\beta_n L^n,$$

evaluated at $L = \frac{1}{1+r}$, and $D\left(\frac{1}{1+r}\right)$ is the corresponding (the AR part of the process) lag polynomial evaluated at $L = \frac{1}{1+r}$.

Let us call (25) a volatility proposition (not a universally acknowledged term). The empirical tests of the orthogonality proposition lead to a puzzle called *excess sensitivity* of consumption (consumption is not orthogonal to lagged income as implied by 23); the empirical tests of the volatility proposition lead to a puzzle called *excess smoothness* of consumption (consumption is less volatile than is implied by 25). In the end of the day we shall see that both excess sensitivity and excess smoothness are manifestations of the same phenomenon — the empirical failure of the permanent income hypothesis. ■